The non-commutative analogue of Korovkin's sets and peak points

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Introduction

The Classical theorems of Korovkin impressed several mathematicians since their discovery for the simplicity and the potential. Positive approximation process play a fundamental role in the approximation theory and it appears in a very natural way in several problems dealing with the approximation of continuous functions and qualitative properties such as monotonicity, convexity, shape preservation and so on. A considerable amount of research extended the Korovkin's theorems to the setting of different function spaces or more general abstract spaces such as Banach spaces, Banach algebras, Banach lattices, C^*-algebras and so on during last fifty years.

Korovkin theorem

The classical approximation theorem due to Korovkin in 1953 unified many existing approximation processes

Theorem

Let $\{\phi_n : n = 1, 2, 3, ...\}$ *be a sequence of positive linear maps from* $C([a, b])$ *to itself. For each function* $g_k(x) = x^k, x \in [a, b],\ k = 0, 1, 2,$ *if*

$$\lim_{n\to\infty} \phi_n(g_k) = g_k \text{ uniformly on } [a, b], k = 0, 1, 2.$$

Then

$$\lim_{n\to\infty} \phi_n(f) = f \text{ uniformly on } [a, b], \text{ for all } f \text{ in } C[a, b].$$

Korovkin set

Definition

A set G in $C([a,b])$ is called a Korovkin set or test set, if for every sequence $\{\phi_n\}$, $n = 1, 2, 3, \ldots$ of positive linear maps from $C([a,b])$ to itself $\lim_{n\to\infty} \phi_n(g) = g$ uniformly on $[a,b]$ for every $g \in G$ implies that $\lim_{n\to\infty} \phi_n(f) = f$ uniformly on $[a,b]$ for every $f \in C([a,b])$.

Korovkin theorem proves that $\{1, x, x^2\}$ is a minimal Korovkin set for $C([a,b])$.

Definition

Let $S \subset C(X)$ containing the constant function 1, where X is a compact Hausdorff space. The Choquet boundary ∂S of S is defined as $\partial S = \{x \in X : \varepsilon_{x|_S}$ has a unique positive linear extension to $C(X)$, where ε_x denotes the evaluation functional defined by $\varepsilon_x(f) = f(x), f \in C(X)\}$.

Theorem

Let S be a subset of $C(X)$ that separates points of X and contains constant function. Then S is a Korovkin set in $C(X)$ if and only if the Choquet boundary $\partial G = X$. Where $G = \text{linear span}(S)$

Peak points

Definition

Let G be a closed subspace of $C(X)$, separating points and containing the identity 1_X of $C(X)$. A point $x_0 \in X$ is a *peak point* of G if there exists a $g \in G$ for which
$g(x_0) = \|g\|, |g(x)| < \|g\|, \; x \neq x_0$.
The set of peak points of G is denoted by $P(G)$.

Theorem

Let G be a closed subspace of $C(X)$, separating points and containing the identity 1_X of $C(X)$; then $P(G) \subset \partial G$.

Theorem

Let G be a closed subspace of $C(X)$, separating points and containing the identity 1_X of $C(X)$; then $\partial G \subset \overline{P(G)}$.

In 1969 Arveson introduced the notion of boundary representation.

Definition

Let S be an operator system in a C^*-algebra A such that $A = C^*(S)$. A representation $\pi : A \to B(H)$ of A is said to have unique extension property (UEP) for S, if the only unital completely positive (UCP) map $\phi : A \to B(H)$ that satisfies $\phi_{|S} = \pi_{|S}$ is $\phi = \pi$ itself.

Definition

Let S be an operator system in a C^*-algebra A such that $A = C^*(S)$. An irreducible representation $\pi : A \to B(H)$ of A is said to be a boundary representation for S if π has unique extension property (UEP) for S.

Hyperrigid set

In 2011 Arveson introduced noncommutative analogue of classical Korovkin set as follows.

Definition

A set G of generators of an abstract C^*-algebra A is said to be *hyperrigid* if for every faithful representation $A \subseteq B(H)$ of A on a Hilbert space and every sequence of unital completely positive maps $\{\phi_n\}$ from $B(H)$ to itself,

$$\lim_{n\to\infty} \|\phi_n(g) - g\| = 0, \quad \forall\, g \in G \Rightarrow \lim_{n\to\infty} \|\phi_n(a) - a\| = 0, \quad \forall\, a \in A.$$

Arveson's hyperrigidity conjecture

Theorem

For every separable operator system S that generates a C^*-algebra $A = C^*(S)$, such that S is hyperrigid if and only if For every nondegenerate representation $\pi : C^*(S) \to B(H_\pi)$ on a separable Hilbert space, $\pi_{|_S}$ has unique extension property.

Theorem

Let S be a separable operator system generating a C^*- algebra A such that $A = C^*(S)$. If S is hyperrigid, then every irreducible representation of A is a boundary representation for S

Hyperrigidity conjecture

Still, hyperrigidity conjecture is not completely resolved. Hyperrigidity conjecture relates boundary representations of C^*-algebras for operator systems with hyperrigidity of operator systems. **It states that, if every irreducible representation of a C^*-algebra A is a boundary representation for a separable operator system $S \subseteq A$, then S is hyperrigid.** Arveson proved the conjecture for C^*-algebras having countable spectrum, while Kleski established the conjecture for all type-I C^*-algebras.

Arveson introduced the notion of *peaking representation*, which is the non-commutative analogue of classical peak points.

Definition

Let S be a separable operator system and let $A = C^*(S)$ is the C^*-algebra generated by S. An irreducible representation $\pi : A \to B(H)$ is said to be a *peaking representation* for S if there is an $n \geq 1$ and an $n \times n$ matrix $[s_{ij}]$ over S such that

$$|| (\pi[s_{ij}]) || > || (\sigma[s_{ij}]) ||$$

for every irreducible representation σ not unitarily equivalent to π.

Arveson and Kleski

Theorem

Let S be an operator system that generates a finite dimensional C^*-algebra $C^*(S)$. An irreducible representation of $C^*(S)$ is a boundary representation for S if and only if it is peaking for S.

Theorem

Let S be a separable operator system that generates a C^*-algebra $C^*(S)$. Then every peaking representation for S is a boundary representation for S.

Weak Choquet boundary

Definition

Let A be a unital C^*-algebra and S be an operator system of A such that $A = C^*(S)$-the C^*-algebra generated by S. An irreducible representation $\pi : A \to B(H_\pi)$ is called *weak boundary representation* for S of A if $\pi_{|_S}$ has a unique UCP map extension of the form $V^*\pi V$, namely π itself, where $V : H_\pi \to H_\pi$ is an isometry.

The set of all weak boundary representations for S of A is called weak Choquet boundary of S and denoted by $\partial_W S$. We can observe that all the boundary representations are weak boundary representations for S. Thus $\partial S \subseteq \partial_W S$.

Example

Let $G =$ linear span(I, S, S^*), where S is the unilateral right shift in $B(H)$ and I is the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by G. We have $K(H) \subseteq A$, $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and the spectrum $\hat{A}$ of A can be identified with $\{Id\} \cup \mathbb{T}$. We know that ε_t is a one dimensional irreducible representation of A for all $t \in \mathbb{T}$, therefore ε_t is a weak boundary representation for G of A for all $t \in \mathbb{T}$. Note that $Id_{|_G}$ has more than one UCP extension from the class $CP(A, Id, H_{Id})$. Observe that $S^* Id(\cdot) S$ is also an extension of $Id_{|_G}$. Therefore, Id is not a weak boundary representation.

Finite reperesentation

Arveson introduced the notion of finite representation in the setting of subalgebras of C^*-algebras. He further proved that any representation π of a subalgebra $\mathscr{A}$ of a C^*-algebra A on a Hilbert space H is *finite representation* if and only if for every isometry V in $B(H)$, the condition $V^*\pi(a)V = \pi(a)$ for all a in $\mathscr{A}$ implies that V is unitary.

Theorem

Let A be a C^*-algebra and S be an operator system in A such that $A = C^*(S)$. Let π be an irreducible representation of A on a Hilbert space H. Then π is a finite representation of S if and only if π is a weak boundary representation for S of A.

Quasi hyperrigidity

Definition

A set S of generators of a C^*-algebra A is said to be quasi hyperrigid, if for every nondegenerate representation π of A on a Hilbert space H_π and for every isometry $V : H_\pi \to H_\pi$ the condition $V^*\pi(s)V = \pi(s)$ for all s in S implies that $V^*\pi(a)V = \pi(a)$ for all a in A.

Here we explore the relation between hyperrigidity and quasi hyperrigidity. It is trivial to see that hyperrigid sets are quasi hyperrigid. However, the converse is not true and hence the notion is strictly weaker. We illustrate an example.

Example

Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{C}$, where $n \geq 3$. Define a unital completely positive map Φ on $M_n(\mathbb{C})$ as given below. Let

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & a_{nn} \end{bmatrix}$$

be arbitrary.

Example

Now define Φ on $M_n(\mathbb{C})$

$$\Phi(M) = \begin{bmatrix} a_{11} & a_{12} & 0 & . & . & . & 0 \\ a_{21} & a_{22} & 0 & . & . & . & 0 \\ 0 & 0 & a_{22} & . & . & . & 0 \\ . & . & . & a_{22} & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a_{22} \end{bmatrix}$$

Now let $M = T$, where $a_{21} = 1$ and all other entries equal to 0. Let $S = span\{I, T, T^*\}$ and $A = C^*(S)$. Consider the sequence of unital completely positive maps $\{\Phi_n\}$ on $C^*(S)$ where $\Phi_n = \Phi$ for all n. Note that for all n, $\Phi_n(s) = s \ \forall \ s \in S$, but $\Phi_n(TT^*) \neq TT^*$. This implies that S is not a hyperrigid set. However, if V is any isometry such that $V^*V = I$, then $VV^* = I$, since A is finite dimensional. Thus, S is quasi hyperrigid, but fails to be a hyperrigid set.

Quasi hyperrigidity

Theorem

Let S be a separable operator system and $A = C^*(S)$. Then S is quasi hyperrigid if and only if for every non-degenerate representation $\pi : A \to B(H_\pi)$ on a separable Hilbert space, $\pi_{|_S}$ has a unique UCP map extension of the form $V^*\pi V$, where $V : H_\pi \to H_\pi$ is an isometry.

Theorem

Let S be a separable operator system generating a C^*-algebra A. If S is quasi hyperrigid, then every irreducible representation of A is a weak boundary representation for S.

Quasi hyperrigidity

Problem

If every irreducible representation of A is a weak boundary representation for a separable operator system $S \subseteq A$, then is S quasi hyperrigid?

Theorem

Let $A = C^*(S)$ be the C^*-algebra generated by a separable operator system S such that A has countable spectrum. If every irreducible representation of A is a weak boundary representation for S then S is quasi hyperrigid.

Weak unique extension property

Definition

Let S be an operator system generating a C^*-algebra A. Let $\pi : A \to B(H_\pi)$ be a representation, then π is said to have weak unique extension property (WUEP) for S if π is the only UCP map extension of $\pi_{|_S}$ of the form $V^*\pi(\cdot)V$, where V is an isometry on H_π.

Theorem

Let S be a separable operator system generating a Type I C^*-algebra A. If every irreducible representation of A is a weak boundary representation for S, then for any UCP map $V^*\pi V : A \to A''$, where $\pi : A \to A''$ is a representation and $V \in A''$ is an isometry such that $V^*\pi(s)V = \pi(s)$ for all $s \in S$ implies that $V^*\pi(a)V = \pi(a)$ for all $a \in A$.

Definition

Let A be a unital C^*-algebra and S be an operator system of A such that $A = C^*(S)$, the C^*-algebra generated by S. An element π of $\hat{A}$ is called a *weak peak point* for S if there exists $s \in S$ such that

(i) $|\langle \pi(s)\xi_\pi, \xi_\pi \rangle| = \|s\|$ for some $\xi_\pi \in H_\pi$ with $\|\xi_\pi\| = 1$,

(ii) $|\langle \sigma(s)\xi_\sigma, \xi_\sigma \rangle| < \|s\|$ for all $\xi_\sigma \in H_\sigma$ with $\|\xi_\sigma\| = 1$,

where σ is any irreducible representation not equivalent to π. We will denote the set of all weak peak points for S by $P_w(S)$.

We observed that the Choquet boundary of an operator system is contained in weak Choquet boundary of it and this inclusion is strict. So it would be interesting to know which weak Choquet boundary points are Choquet boundary points of an operator system. The following theorem gives partial answer to this query.

Theorem

Let S be an operator system in a C^*-algebra $A = C^*(S)$. If $\pi \in \hat{A}$ is a weak peak point for S, π is a weak boundary representation for S and $\pi_{|_S}$ is pure, then π is a boundary representation for S.

Examples

Example 1

Let the Volterra integration operator V acting on the Hilbert space $H = L^2[0,1]$ is given by

$$Vf(x) = \int_0^x f(t)dt, \qquad f \in L^2[0,1].$$

It is well known that V generates the C^*-algebra $K = K(H)$ of all compact operators. Let $S = span\left(V, V^*, V^2, V^{2*}\right)$ and S is hyperrigid. Let $\tilde{S} = S + \mathbb{C}\cdot\mathbf{1}$ be an operator system generating the C^*-algebra $\tilde{A} = K + \mathbb{C}\cdot\mathbf{1}$. The irreducible representations of $\tilde{A}$ are π and ρ given by

$$\pi(T + \lambda\mathbf{1}) = T, \quad \text{for} \quad T \in K,\ \lambda \in \mathbb{C}$$

$$\rho(T + \lambda\mathbf{1}) = \lambda, \quad \text{for} \quad T \in K,\ \lambda \in \mathbb{C}$$

Example 1

In fact these are the only two irreducible representations upto unitary equivalence. $\tilde{S}$ is a hyperrigid operator system implying that π and ρ are boundary representations for $\tilde{S}$ of $\tilde{A}$. Also, $\tilde{S}$ is quasi hyperrigid and therefore π, ρ are weak boundary representations for $\tilde{S}$.
Let $V + V^* \in \tilde{S}$ be the projection on the space of constants and let the constant function $1 \in L^2[0,1]$, $||1|| = 1$

$$|\langle \pi(V + V^*)1, 1\rangle| = 1 = ||V + V^*||.$$

For all $\xi_\rho \in \mathbb{C}$, $||\xi_\rho|| = 1$

$$|\langle \rho(V + V^*)\xi_\rho, \xi_\rho\rangle| = |\langle 0\xi_\rho, \xi_\rho\rangle| = 0 < ||V + V^*||.$$

Therefore π is a weak peak point.

Example 1

Let $\mathbf{1} \in \tilde{S}$ and $1 \in \mathbb{C}$, $||1|| = 1$

$$|\langle \rho(\mathbf{1})1, 1\rangle| = 1 = ||\mathbf{1}||.$$

For all $\xi_\pi \in L^2[0,1]$, $||\xi_\pi|| = 1$

$$|\langle \pi(\mathbf{1})\xi_\pi, \xi_\pi\rangle| = |\langle 0\xi_\pi, \xi_\pi\rangle| = 0 < ||\mathbf{1}||.$$

Hence ρ is a weak peak point. Also, π and ρ restricted to $\tilde{S}$ are pure.

Example 2

Let $G = span(I, S, S^*, SS^*)$, where S is the unilateral right shift in $B(H)$ and I the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by G. We have, $K(H) \subseteq A$. $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and the spectrum $\hat{A}$ of A can be identified with $\{Id\} \cup \mathbb{T}$. Since S is an isometry, G is hyperrigid and this will imply that all the irreducible representations of A are boundary representations for S. Clearly G is quasi hyperrigid, so all the irreducible representations are weak boundary representations for S.

Examples

Example 2

Now we prove that identity representation Id of A is a weak peak point for G. Let $e_1 = (1,0,0...,0)$ and let $E = I - SS^* \in G$ be the rank one projection. We have $|\langle Id(E)e_1, e_1\rangle| = 1 = ||E||$ and for any irreducible representation π which is not equivalent to identity, $\pi(E) = 0$. So we have $|\langle \pi(E)\eta, \eta\rangle| = 0 < ||E||$ for all unit vectors $\eta \in H_\pi$. This proves that Id is a weak peak point. Also, $Id_{|_G}$ is pure.

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Thank You